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THEORY OF A GYROVERTICAL COMPASS

by

A. Yu. Ishlinskiy

FOREIGN TECHNOLOGY DIVISION

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В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
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М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ë in Russian, transliterate as yë or ë.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

0926

THEORY OF A GYROVERTICAL COMPASS

A. Yu. Ishlinskiy (Kiev)

1. Below we will give a strict account of the theory of a gyrovertical compass whose sensor is a device similar to the so-called gyrosphere of the "new Anschuetz" gyroscopic compasses [1, 2].

This sensor can be considered to be a set of two gyroscopes. The axes of the housings of these gyroscopes are parallel to each other, while their journal bearings are rigidly fastened to the same framework, which will henceforth also be called the gyroscopic frame or simply the gyroframe (Fig. 1). In a two-gyroscope compass, this frame is surrounded by a spherical shell and is submerged in a liquid, which provides extremely perfect suspension of the framework

with almost no friction (Fig. 2).

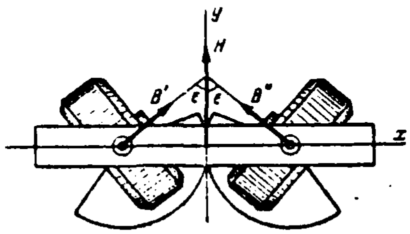

Fig. 1.

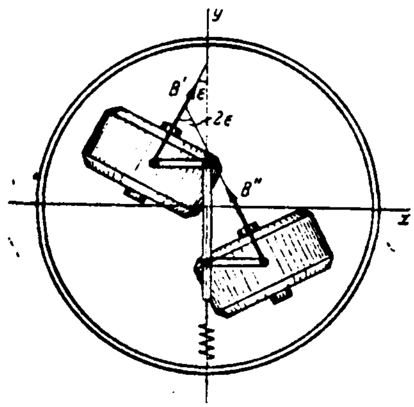

Fig. 2.

We will assume that the center of suspension of the framework moves over a certain sphere S with radius R which surrounds the Earth, while the force of gravity of the framework to the Earth is reduced to the single force F applied to the center of gravity of the framework (together with the gyroscopes) and directed toward the center of the sphere.

We will consider that the sphere S does not participate in the Earth's rotation and its orientation relative to stationary stars does not change. In the future, the translational motion of the sphere is insignificant, since it takes place with vanishingly small acceleration. Therefore, we can consider the center of the sphere S

to be fixed.

As we will show later, it is very easy to investigate the movement of the sensor relative to fixed sphere S [3].

We will disregard the forces of friction in the suspension of the frame itself and in the bearings of the axes of the gyroscope housings, as well as unavoidable assembly defects, e.g., the presence of axial and radial gaps in the bearings and residual inequilibrium of the gyroscopes around the axes of their housings.

We will assume that like for the sensor of a two-gyroscope compass, a special gear drive (or four-component mechanism) turns the gyroscope housings relative to the framework to different sides at angles which can be considered to be equal (Fig. 1 and 2).

2. According to the precession (so-called elementary) theory of gyroscopic phenomena, we will consider the total kinetic moment H of the entire gyroscopic framework to be equal to the geometric sum of the intrinsic kinetic moments of the gyroscopes B' and B", which have the same values. We will use 2ε to designate the angle between the axes of natural rotation of the gyroscopes (Fig. 1). Then

$$H = 2B\cos\varepsilon \quad (B = B' = B'') \quad (1)$$

The total kinetic moment H is directed along the bissectrix of angle 2ε. Because of the presence of the aforementioned gear drive, the vector H does not change position relative to the framework.

We will bind coordinate system xyz with its origin in the center of suspension to the framework, with the y-axis directed parallel to the vector H, and the z-axis parallel to the axes of the gyroscope housings. The position of the x-axis is thereby uniquely defined (Fig. 3).

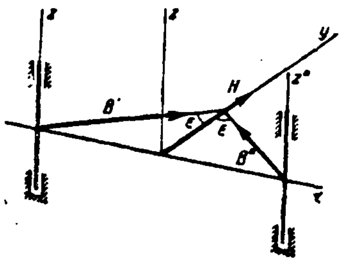

Fig. 3.

We will use $\omega_x, \omega_y, \omega_z$ to designate the projections of the angular velocity of the framework relative to sphere S (or, equivalently, relative to a counting system bound to stationary

stars) onto the axes of this coordinate system. The expressions

$$\frac{dH_x}{dt}+\omega_y H_z-\omega_z H_y, \quad \frac{dH_y}{dt}+\omega_z H_x-\omega_x H_z, \quad \frac{dH_z}{dt}+\omega_x H_y-\omega_y H_x \quad (2)$$

are the projections of the velocity of the end of the vector of the kinetic moment of the gyroscopic frame H onto the x-, y- and z-axes, with the assumption that the origin of the vector is fixed. According to the known theorem of mechanics, these projections are equal to the sums of the moments of the forces acting on the framework around the same axes, respectively. We will designate these sums as M_x, M_y and M_z.

The kinetic moment of the framework is directed along the y-axis. Therefore,

$$H_x=0, \; H_y=H=2B\cos\varepsilon, \; H_z=0.$$

As a result, we will have the three equations:

$$-\omega_z H=M_x, \quad \frac{dH}{dt}=M_y, \quad \omega_x H=M_z \quad (3)$$

In the case in question, the gyroscopic framework is a mechanical system with four degrees of freedom. Therefore, in order to completely describe the laws of its movement, we should add another equation containing the projection of the angular velocity ω_y to equations (3). For this purpose, we will point out that the projections of the velocities of the ends of the intrinsic kinetic

moments of the gyroscopes onto the z-axis are expressed by the formulae

$$\omega_x B_y' - \omega_y B_x' = M_z', \qquad \omega_x B_y'' - \omega_y B_x'' = M_z'' \qquad (4)$$

Here M_z' and M_z'' are the sums of the moments of the forces acting on the housings of each of the gyroscopes around the axes of these housings, and the values B_x', B_y', B_x'', B_y'' are the projections of the intrinsic kinetic moments of the gyroscopes of the frame B' and B" onto the x- and y-axes, respectively. Obviously (Fig. 3)

$$B_x' = -B_x'' = B\sin\varepsilon, \qquad B_y' = B_y'' = B\cos\varepsilon \qquad (5)$$

According to the precession (elementary) theory of gyroscopes, only the intrinsic kinetic moments of the gyroscope rotors are considered in the equations of movement of the mechanical system. The remaining kinetic moments and the changes corresponding to them are not considered. Thus, we should consider the forces directly applied to the framework to be balanced relative to each other. In particular, we will have

$$M_z - M_z' - M_z'' = 0 \qquad (6)$$

Here the moments $-M_z'$ and $-M_z''$ are the counteraction of the gyroscope housings, to the axes of which moments M_z' and M_z'' are applied by the framework. Substituting the expressions for moments M_z' and M_z'' in relationship (6) according to formulae (4) and considering

equations (1) and (5), we will again obtain the third equation in (3).

Now we will formulate the difference of moments M_z' and M_z'' and designate it as N. According to formulae (4) and (5), we will have

$$N = M_z' - M_z'' = -\omega_y 2B \sin \varepsilon \qquad (7)$$

Moment N can be created by a special spring device (Fig. 2). In this case, it is a function of angle ε.

Thus, the movement of the gyrcscopic frame is determined according to relationships (3), (7), and formula (1) by the following four equations

$$\begin{gathered} -\omega_z 2B \cos \varepsilon = M_x, \qquad \omega_x 2B \cos \varepsilon = M_z \\ \frac{d}{dt} 2B \cos \varepsilon = M_y', \qquad -\omega_y 2B \sin \varepsilon = N \end{gathered} \qquad (8)$$

3. It turns out that the parameters characterizing the gyroscopic framework and, in particular, the form of the dependence of moment N on angle ε, can be selected so that when the specified initial conditions are satisfied, the z-axis will always be normal to sphere S, as if the suspension point of the framework did not move over it.

In order to prove this interesting theorem of theoretical

mechanics, we should use equations (8) and explain the circumstances under which they can be satisfied identically.

We will introduce a certain moving coordinate system $\xi^*\eta^*\zeta^*$, whose origin is located at the suspension point of the gyroscope, while the ξ^*-, η^*- and ζ^*-axes are oriented toward stationary stars. Equations (8) are precisely the equations of movement of the gyroscopic framework relative to this coordinate system. Along with the force of gravity to the Earth's center and the reaction of the suspension, the forces acting on the framework should include the forces of inertia of its translational movement, together with the progressively moving coordinate system $\xi^*\eta^*\zeta^*$. The latter are reduced to the single force Q applied to the center of gravity of the framework. The projections of this force onto the axes of coordinates x, y and z are:

$$Q_x = -mw_x, \qquad Q_y = -mw_y, \qquad Q_z = -mw_z \tag{9}$$

Here m is the mass of the framework together with the housings and rotors of its gyroscopes, w_x, w_y and w_z are the projections of the acceleration of the suspension point of the framework as it moves over sphere S onto the axes of the coordinate system.

According to the known kinematic formulae [4]

$$\begin{aligned} w_x &= \frac{dv_x}{dt} + \omega_y v_z - \omega_z v_y \\ w_y &= \frac{dv_y}{dt} + \omega_z v_x - \omega_x v_z \\ w_z &= \frac{dv_z}{dt} + \omega_x v_y - \omega_y v_x \end{aligned} \qquad (10)$$

where v_x, v_y, v_z are the projections of the velocity of the suspension point, and ω_x, ω_y, ω_z are the projections of the angular velocity of the framework itself and, consequently, also coordinate system x, y, z relative to sphere S. Since, by assumption, the suspension point moves over sphere S and the z-axis must be normal to this sphere, [3]

$$v_x = \omega_y R, \qquad v_y = -\omega_x R, \qquad v_z = 0 \qquad (11)$$

Using formulae (9), (10) and (11), the projections of force Q onto the axis of coordinate system xyz can now be represented as

$$Q = -mR\left(\frac{d\omega_y}{dt} + \omega_z\omega_x\right), \quad Q_y = -mR\left(-\frac{d\omega_x}{dt} + \omega_z\omega_y\right)$$
$$Q_z = -mR(-\omega_x^2 - \omega_y^2) \qquad (12)$$

Let the center of gravity of the framework be located on the negative part of the z-axis at a distance l from the suspension point. In this case, the force of gravity is directed along the z-axis and, therefore, its moment relative to the suspension point is equal to zero. The same is true of the component of the force of inertia of translational motion and the force of the coupling

reaction. Therefore, it suffices to find the moments of forces Q_x and Q_y relative to the x-, y- and z-axes in order to determine moments M_x, M_y and M_z. As a result, we obtain the expressions

$$M_x = lQ_y, \qquad M_y = -lQ_x, \qquad M_z = 0 \tag{13}$$

Substituting these expressions in equations (8) and considering formula (11), we obtain the following equations:

$$\begin{aligned} &-\omega_z 2B\cos\varepsilon = mlR\left(\frac{d\omega_x}{dt} - \omega_y\omega_z\right), \qquad \omega_x 2B\cos\varepsilon = 0 \\ &\frac{d}{dt}(2B\cos\varepsilon) = mlR\left(\frac{d\omega_y}{dt} + \omega_x\omega_z\right) \qquad -\omega_y 2B\sin\varepsilon = N \end{aligned} \tag{14}$$

which must be satisfied identically.

According to the third expression (if we do not consider the exceptional case when $\varepsilon = {}^1/_2\pi$), we will have

$$\omega_x = 0 \tag{15}$$

Now it is not hard to see that the first two equations of (14) are satisfied if the condition

$$2B\cos\varepsilon = H = mlR\,\omega_y \tag{16}$$

is observed.

Using this condition to eliminate the value ω_y from the fourth equation of (14), we arrive at the relationship

$$N = -\frac{4B^2}{mlR}\cos\varepsilon\sin\varepsilon \tag{17}$$

which determines the unknown form of the dependence of moment N on angle ε.

Relationships (11), (15), (16) and (17) make it possible to explain what the initial conditions of the movement of the gyroscopic frame must be in order for the described movement to be possible. According to relationship (15) and the second formula in (11), we will have

$$v_y = 0 \tag{18}$$

Therefore, at the initial point in time, the x-axis, which is bound to the gyroscopic frame, must be directed along the tangent to the trajectory of the suspension point during its movement over sphere S (Fig. 4). The x-axis will only be tangent to the above trajectory at all times when the remaining initial conditions, which will be explained below, are also satisfied.

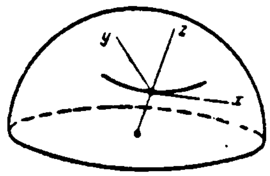

Fig. 4.

According to the first formula in (11) and relationships (15) and (16), we obtain the equation

$$2B \cos \varepsilon = mlv \qquad (19)$$

where v is the velocity of the framework's suspension point relative to sphere S. Therefore, if the suspension point was v_0 at the initial point in time, the initial value or angle ε_0 should be determined by the formula

$$\cos \varepsilon_0 = \frac{mlv_0}{2B} \qquad (20)$$

In the future, based on the second equation of (14) and relationship (15), formula (19) remains valid throughout the time of movement during random movement of the suspension point. Finally, at the initial point in time the z-axis, which is parallel to the axes

of the gyroscope housings, should be normal to sphere S. When these conditions are satisfied, according to relationship (17), moment N will be such that on the basis of the fourth equation in (14), the angular velocity component of the framework ω_ψ determined by it satisfies the first equation in (11).

At the initial point in time, if conditions (18) and (20) are satisfied with a small error and the z-axis is deflected from the normal to the surface of S by a small angle, the movement of the framework can be examined by studying small oscillations around movement during which the initial conditions are satisfied precisely. We will return to this problem in §6.

4. Now we will turn our attention to studying the movement of the gyroframe relative to the Earth, considering it to be a sphere with radius R and considering that all of the conditions in §3 are satisfied precisely.

We will introduce the moving coordinate system $\xi\eta\zeta$, the ξ-axis of which is directed along the tangent to the parallel to the east, the η-axis - along the tangent to the meridian to the north, and the ζ-axis along the Earth's radius upward. We will place the origin of coordinates at the framework suspension point.

We will use U to designate the Earth's angular velocity, φ - the latitude of the location (strictly speaking, geocentric), and V_E and V_N - the eastern and northern components of the velocity of the origin of system $\xi\eta\zeta$ relative to the Earth. The projections of the velocity of this point relative to sphere S onto the ξ- and η-axes are represented as follows:

$$v_\xi = V_E + UR\cos\varphi, \qquad v_\eta = V_N \tag{21}$$

Now, using formula (11), after first replacing the letters x and y in them by ξ and η, respectively, we obtain the known formulae

$$u_\xi = -\frac{V_N}{R}, \qquad u_\eta = \frac{V_E}{R} + U\cos\varphi \tag{22}$$

for the projections of the angular velocity of the triangle $\xi\eta\zeta$ relative to coordinate system $\xi^*\eta^*\zeta^*$, which is oriented according to stationary stars. As we know, the projection of the above angular velocity onto the ζ-axis is expressed by the formula [1]

$$u_\zeta = \frac{V_E}{R}\,\mathrm{tg}\,\varphi + U\sin\varphi \tag{23}$$

We will use ϑ to designate the angle between the y- and η-axes, counting the positive direction of this angle as is shown in Fig. 5. It is not hard to see that here the projections of the angular

velocity of coordinate system xyz, which is bound to the gyroframe, onto the x-, y- and z-axes has the form:

$$\omega_x = u_\xi \cos\vartheta - u_\eta \sin\vartheta, \quad \omega_y = u_\xi \sin\vartheta + u_\eta \cos\vartheta, \quad \omega_z = u_\zeta - \frac{d\vartheta}{dt} \qquad (24)$$

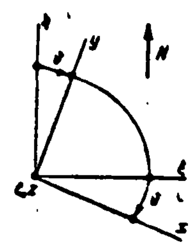

Fig. 5.

According to the laws of movement of the gyroframe given in §3, we must plug $\omega_x = 0$ into the first of these formulae. Furthermore, if we replace the values u_ξ and u_η in them by their representation according to formulae (22), we will have

$$\operatorname{tg}\vartheta = -\frac{V_N}{RU\cos\varphi + V_E} \qquad (25)$$

Thus, the y-axis, which is bound to the gyroframe, is deflected from the direction to the north by angle ϑ, which is determined by formula (25). This agrees with the known formula for the so-called velocity deviation of a gyroscopic compass.

Now we will direct our attention to the relationship (19) between the kinetic moment of the gyroframe and the rate of movement of its suspension point over sphere S. Considering formulae (21) and (1), this relationship can be represented[1] as follows:

$$H = 2B\cos\varepsilon = ml\sqrt{(RU\cos\varphi + V_E)^2 + V_N^2} \qquad (26)$$

Footnote: [1]This very relationship was obtained earlier by V. G. Zheleznov when refining the known Schuler condition $H = mlRU\cos\phi$ in the approximate theory of gyroscopic compasses. When the Schuler condition is satisfied with a certain degree of approximation, the deviation of the gyroscopic compass does not depend on the former values of the velocity of the ship and its acceleration, but is mainly determined by formula (25). Other researchers (E. I. Sliv, Ya. N. Roytenberg) refined the Schuler condition (for the case of high latitudes) by considering only one eastern component of the velocity of the ship V_E, and it was represented in the form $H = ml(RU\cos\phi + V_E)$. End footnote

The above information shows that relationship (26) acutally provides precise adherence of the deviation of the gyroscopic compass

to law (25), as long as the dependence of the moment N on the angle ε in formula (17) is satisfied and, furthermore, the initial conditions in §3 are observed. We will point out that in this case, the z-axis, which bound to the gyroscopic framework, is directed towards the center of the Earth. Therefore, it forms a certain small angle with the vertical which depends on the latitude of the location.

5. We will construct the Darboux triangle $x^0y^0z^0$ with the apex at the suspension point of the gyroframe. We will direct its x^0-axis along the velocity vector of the suspension point of the framework relative to sphere S and the z^0-axis normal to the sphere. Here the direction of the y^0-axis is completely defined. If the initial conditions of the movement of the gyroframe given in §3 are observed, the x-, y- and z-axes, which are bound to the gyroframe, will always coincide with the x^0-, y^0- and z^0-axes during random movement of the triangle over surface S.

We will consider the general case of the initial conditions of the gyroframe, and we will construct the equations of its movement relative to triangle $x^0y^0z^0$.

We will define the position of coordinate system xyz relative to triangle $x^0y^0z^0$ as shown in Fig. 6 and 7 by the three angles α, β and

γ. Angle α determines the rotation around the z-axis, which coincides with the z^0-axis of auxiliary coordinate system $x'y'z'$ relative to system $x^0y^0z^0$. In turn, angle β is the angle of rotation around the x''-axis (or, analogously, the x'-axis) of the other auxiliary coordinate system $x''y''z''$ relative to the first, i.e., the system $x'y'z'$. Finally, angle γ is the angle of rotation around the y-axis (it is also the y''-axis) of coordinate system xyz relative to system $x''y''z''$. The table of cosines of the angles between coordinate systems xyz and $x^0y^0z^0$ is:

	x^0	y^0	z^0	
x	$\cos\alpha\cos\gamma-\sin\alpha\sin\beta\sin\gamma$	$\sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma$	$-\cos\beta\sin\gamma$	
y	$-\sin\alpha\cos\beta$	$\cos\alpha\cos\beta$	$\sin\beta$	(27)
z	$\cos\alpha\sin\gamma+\sin\alpha\sin\beta\cos\gamma$	$\sin\alpha\sin\gamma-\cos\alpha\sin\beta\cos\gamma$	$\cos\beta\cos$	

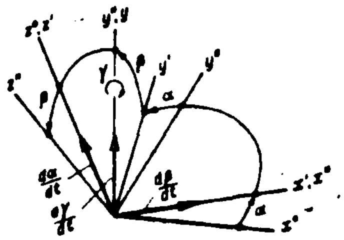

Fig. 6.

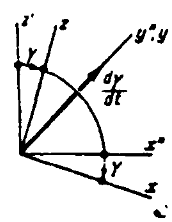

Fig. 7.

In order to obtain the projection of the angular velocity of coordinate system xyz onto its same axes, we should take the sum of

the projections of the angular velocity of triangle $x^0y^0z^0$ onto these axes, as well as the relative angular velocities: $d\alpha/dt$ of coordinate system $x'y'z'$ relative to $x^0y^0z^0$, $d\beta/dt$ - of system $x''y''z''$ relative to $x'y'z'$ and, finally $d\gamma/dt$ of coordinate system xyz relative to system $x''y''z''$.

The angular velocity vector $d\alpha/dt$ is directed along the z^0-axis, and the angular velocity vector $d\gamma/dt$ - along the y''-axis. In turn, the angular velocity vector $d\beta/dt$ is in the same direction as the x'-axis. The x'-axis, which coincides with the x''-axis, is the intersection of the coordinate planes x^0y^0 and zx. It forms angles γ, $^1/_2\pi$ and $^1/_2\pi - \gamma$, respectively, with the axes of the coordinate system.

Considering these circumstances and the cosine table (27), we arrive at the following expressions for the unknown projections:

$$\begin{aligned}\omega_x &= \omega_{x^0}(\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma) + \omega_{y^0}(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma) + \\ &\quad + \left(\omega_{z^0} + \frac{d\alpha}{dt}\right)(-\cos\beta\sin\gamma) + \frac{d\beta}{dt}\cos\gamma \qquad (28)\\ \omega_y &= \omega_{x^0}(-\sin\alpha\cos\beta) + \omega_{y^0}\cos\alpha\cos\beta + \left(\omega_{z^0} + \frac{d\alpha}{dt}\right)\sin\beta + \frac{d\gamma}{dt}\\ \omega_z &= \omega_{x^0}(\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma) + \omega_{y^0}(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) + \\ &\quad + \left(\omega_{z^0} + \frac{d\alpha}{dt}\right)\cos\beta\cos\gamma + \frac{d\beta}{dt}\sin\gamma\end{aligned}$$

Here ω_{x^0}, ω_{y^0} and ω_{z^0} designate the projections of the angular velocity of triangle $x^0y^0z^0$ onto its intrinsic axis. According to

formula (11), we will have

$$\omega_{x^0} = -\frac{v_{y^0}}{R}, \quad \omega_{y^0} = \frac{v_{x^0}}{R} \tag{29}$$

where v_{x^0} and v_{y^0} are the projections of the velocity of the apex of triangle $x^0y^0z^0$ onto its x^0- and y^0-axes. However,

$$v_{y^0} = 0 \tag{30}$$

since the velocity of the apex of the triangle, by assumption, is directed along the x^0-axis. Considering formula (29) and equation (30), according to expressions (28) we obtain the following formulae for the projections of the angular velocity of the gyroframe to the axes bound to it:

$$\begin{aligned}
\omega_x &= \frac{v}{R}(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma) + \left(\omega + \frac{d\alpha}{dt}\right)(-\cos\beta\sin\gamma) + \frac{d\beta}{dt}\cos\gamma \\
\omega_y &= \frac{v}{R}\cos\alpha\cos\beta + \left(\omega + \frac{d\alpha}{dt}\right)\sin\beta + \frac{d\gamma}{dt} \\
\omega_z &= \frac{v}{R}(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) + \left(\omega + \frac{d\alpha}{dt}\right)\cos\beta\cos\gamma + \frac{d\beta}{dt}\sin\gamma
\end{aligned} \tag{31}$$

In these formulae

$$v = v_{x^0}, \qquad \omega = \omega_{z^0} \tag{32}$$

are the velocity of the apex of the triangle relative to sphere S and its angular velocity component along the normal to this sphere, respectively. Formulae (31) should be plugged into the left sides of equations (18) of the movement of the gyroscopic framework.

Now we will calculate the right sides of these same equations. According to the assumption made in §3, the force of gravity F of the gyroscopic framework to the Earth is directed to the center of sphere S and applied to the center of gravity of the framework. With a high degree of precision we can consider it to be parallel to the z^0-axis. Then, according to the cosine table (27), its projections onto the x-, y- and z-axes, which are bound to the gyroframe, are represented by the expressions

$$F_x = F\cos\beta\sin\gamma, \qquad F_y = -F\sin\beta, \qquad F_z = -F\cos\beta\cos\gamma \quad (33)$$

In order to calculate similar projections of the force of inertia Q of the translational movement (see §3), we should first use formulae (12), representing them, with consideration of equations (29), (30) and (32), as

$$Q_{x^*} = -m\frac{dv}{dt}, \qquad Q_{y^*} = -m\omega v, \qquad Q_{z^*} = m\frac{v^2}{R} \qquad (34)$$

We will point out that formulae (34) can also be obtained indirectly, if we consider that the expressions

$$w_t = \frac{dv}{dt}, \qquad w_p = \frac{v^2}{\rho_g}, \qquad w_n = -\frac{v^2}{R} \qquad (35)$$

are projections of the acceleration of a point moving over a sphere on the axis of the Darboux triangle bound to the trajectory of this point. Here we must consider that the radius of geophysical curvature of the trajectory ρ_g, the angular rotation rate ω of the Darboux triangle around the normal to the sphere (around the z^0-axis), and the velocity of its apex v are connected by the relationship

$$v = \omega \rho_g \tag{36}$$

Using formulae (34) and the cosine table (27), we obtain the following expressions for the projections of the force of inertia Q onto the x-, y- and z-axes, which are bound to the gyroframe:

$$\begin{gathered} Q_x = -m\frac{dv}{dt}(\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma) - \\ - m\omega v(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma) + m\frac{v^2}{R}(-\cos\beta\sin\gamma) \qquad (37) \\ Q_y = -m\frac{dv}{dt}(-\sin\alpha\cos\beta) - m\omega v\cos\alpha\cos\beta + m\frac{v^2}{R}\sin\beta \\ Q_z = -m\frac{dv}{dt}(\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma) - \\ - m\omega v(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) + m\frac{v^2}{R}\cos\beta\cos\gamma \end{gathered}$$

In coordinate system xyz, the center of gravity of the gyroscopic framework has the following coordinates: $x_c = y_c = 0$, $z_c = -l$.

Therefore, the unknown moments M_x, M_y, M_z of the forces acting on the gyroscopic framework can be represented by the formulae

$$M_x = l(F_y + Q_y) + M_x^*, \quad M_y = -l(F_x + Q_x) + M_y^*, \quad M_z = M_z^* \tag{38}$$

where M_x^*, M_y^*, M_z^* are the moments of any other forces (besides the forces of gravity and inertia) which are also applied to the gyroscopic framework relative to the x-, y- and z-axes.

As for the reaction forces of the suspension point of the framework or the forces of the pressure of the fluid on the gyrosphere (for a "Kurs" type gyroscopic compass), their moments relative to the x-, y- and z-axes are all equal to zero. Replacing the values F_x, F_y, Q_x and Q_y in formulae (38) by the expressions according to equations (33) and (37), we will have

$$\begin{gathered} M_x = l\left[-m\frac{dv}{dt}(-\sin\alpha\cos\beta) - m\omega v\cos\alpha\cos\beta + \left(m\frac{v^2}{R} - F\right)\sin\beta\right] + M_x^* \\ M_y = -l\left[-m\frac{dv}{dt}(\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma) -\right] \\ - m\omega v(\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma) + \left(m\frac{v^2}{R} - F\right)(-\cos\beta\sin\gamma)\Big] + M_y^* \\ M_z = M_z^* \end{gathered} \tag{39}$$

Like (31), these expressions should be plugged into equations (14). As a result, we obtain the following equations of the movement of the gyroscopic framework relative to the Darboux triangle:

$$
\begin{gathered}
-\left[\frac{v}{R}(\sin\alpha\sin\gamma-\cos\alpha\sin\beta\cos\gamma)+\left(\omega+\frac{d\alpha}{dt}\right)\cos\beta\cos\gamma+\right.\\
\left.+\frac{d\beta}{dt}\sin\gamma\right]2B\cos\varepsilon=ml\frac{dv}{dt}\sin\alpha\cos\beta-ml\omega v\cos\alpha\cos\beta+\\
+\left(ml\frac{v^2}{R}-lF\right)\sin\beta+M_x^*\\
\frac{d}{dt}2B\cos\varepsilon=ml\frac{dv}{dt}(\cos\alpha\cos\gamma-\sin\alpha\sin\beta\sin\gamma)+ml\omega v(\sin\alpha\cos\gamma+\\
+\cos\alpha\sin\beta\sin\gamma)+\left(ml\frac{v^2}{R}-lF\right)\cos\beta\sin\gamma+M_y^*\\
\left[\frac{v}{R}(\sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma)-\left(\omega+\frac{d\alpha}{dt}\right)\cos\beta\sin\gamma+\right.\\
\left.+\frac{d\beta}{dt}\cos\gamma\right]2B\cos\varepsilon=M_z^* \qquad (40)\\
-\left[\frac{v}{R}\cos\alpha\cos\beta+\left(\omega+\frac{d\alpha}{dt}\right)\sin\beta+\frac{d\gamma}{dt}\right]2B\sin\varepsilon=N(\varepsilon)
\end{gathered}
$$

Equations (40) are valid for any gyroscopic framework. Furthermore, if we set

$$M_x^*=M_y^*=M_z^*=0 \qquad (41)$$

in them, and if we use formula (17) for moment $N(\varepsilon)$, they will be related to the movement of our special gyroframe, the properties of which are given in §3. In this case, as one would expect, equations (45) are satisfied precisely, if we set

$$\alpha=\beta=\gamma=0 \qquad (42)$$

in them and if we find angle ε from relationship (19).

In this case, functions $v = v(t)$ and $\omega = \omega(t)$, which give the movement of the suspension point of the gyroframe over surface S, can be completely arbitrary. When relationships (42) and (19) are satisfied, the gyroscopic framework moves as was described in §§3 and 4. That is to say, the kinetic moment of the framework, which is in the direction of the y-axis, remains perpendicular to the velocity vector of the suspension point relative to sphere S during any movement of this point. The z-axis, which is parallel to the axes of the housings of the gyroscope frameworks, continuously passes through the center of the sphere.

6. Equation (40) of the movement of the gyroscopic framework around the axes $x^0y^0z^0$ of the Darboux triangle, which is bound to the trajectory of movement of the suspension point, is too complex for studying the movement of the framework in the most general case. Therefore, we will limit ourselves to studying small movements of the framework relative to this triangle. On the basis of this, we will retain only the first-order terms relative to angles α, β and γ and their time derivatives in equations (40).

Keeping equations (41) and formula (17) in mind, we will have

$$-\left(\frac{d\alpha}{dt}-\frac{v}{R}\beta+\omega\right)2B\cos\varepsilon=l\left[m\frac{dv}{dt}\alpha-\left(F-m\frac{v^2}{R}\right)\beta-m\omega v\right]$$
$$\frac{d}{dt}(2B\cos\varepsilon)=l\left[mv\omega\alpha-\left(F-m\frac{v^2}{R}\right)\gamma+m\frac{dv}{dt}\right]$$
$$\left(\frac{d\beta}{dt}+\frac{v}{R}\alpha-\omega\gamma\right)2B\cos\varepsilon=0 \qquad (43)$$
$$-\left(\frac{d\gamma}{dt}+\frac{v}{R}+\omega\beta\right)2B\sin\varepsilon=-\frac{4B^2}{mlR}\cos\varepsilon\sin\varepsilon$$

These equations should be considered to be the equations of the disturbed movement of the gyroscope framework relative to the initial movement, during which the angles α, β, γ and ε are determined by relationships (42) and (19). Subsequently designating angle ε of the undisturbed movement as ε^0, in equations (43) we will have

$$\varepsilon=\varepsilon^0+\delta \qquad (44)$$

where δ is a small value of the same order of magnitude as angles α, β and γ. Preserving the first-order terms already related to all four angles α, β, γ and γ in equations (43), and considering that according to condition (19), we should set

$$2B\cos\varepsilon^0=mlv \qquad (45)$$

we arrive at the system of equations of the disturbed movement of the gyroscope framework

$$-mlv\frac{d\alpha}{dt}-ml\frac{dv}{dt}\alpha+lF\beta=-\omega 2B\sin\varepsilon^0\delta \qquad (46)$$
$$\frac{d\beta}{dt}+\frac{v}{R}\alpha=\omega\gamma, \qquad \frac{d\gamma}{dt}+\frac{2B\sin\varepsilon^0}{mlR}\delta=-\omega\beta$$
$$-\frac{d}{dt}(2B\sin\varepsilon^0\delta)+l\left(F-\frac{mv^2}{R}\right)\gamma=\omega mlv\alpha$$

If the velocity of the suspension point of the gyroscope framework v and angular velocity ω of the Darboux triangle around the normal to sphere S are constant, system (46) will become a homogeneous system of linear differential equations with constant coefficients. Approximately assuming that

$$F-\frac{mv^2}{R}\approx F\approx mg \qquad (47)$$

where g is the acceleration of the force of gravity, we arrive at the characteristic equation of system (46), the roots of which are

$$\pm i(\nu+\omega), \qquad \pm i(\nu-\omega) \qquad (48)$$

Here $\nu=\sqrt{g/R}$ is the frequency corresponding to the Schuler period

$$T=2\pi\sqrt{R/g} \qquad (49)$$

According to the approximate theory of a three-dimensional gyrocompass proposed by Gekkeler [2], in this case two noninterrelated oscillations, each of which has a Schuler period,

should take place: the first is related to variables α and β, and the second - to variables γ and δ. The above information shows that the Gekkeler theory contains significant inaccuracies, although it generally leads to correct relationships of type (17) and (19) for selecting the characteristic parameters of the gyrocompass sensor, whose plane xy remains horizontal (according to Gekkeler - almost horizontal) during any maneuvers of the ship.

If approximate formula (47) remains valid, system of equations (46) can also be integrated with variable values of v and ω, i.e., during arbitrary movement of the suspension point over sphere S.

Actually, using equations (47) and (48), equations (46) can be represented as

$$\frac{d}{dt}\frac{v\alpha}{\sqrt{gR}} - \nu\beta = \omega\frac{2B\sin\varepsilon^0\delta}{ml\sqrt{gR}}, \quad \frac{d\beta}{dt} + \nu\frac{v\alpha}{\sqrt{gR}} = \omega\gamma \tag{50}$$

$$\frac{d\gamma}{dt} + \nu\frac{2B\sin\varepsilon^0}{ml\sqrt{gR}}\delta = -\omega\beta, \quad \frac{d}{dt}\left(\frac{2B\sin\varepsilon^0}{ml\sqrt{gR}}\delta\right) - \nu\gamma = -\omega\frac{v\alpha}{\sqrt{gR}}$$

Now we will introduce two new complex-valued functions of the real argument t according to the formulae

$$\varkappa(t) = \frac{v\alpha}{\sqrt{gR}} + i\beta, \quad \mu(t) = \gamma - i\frac{2B\sin\varepsilon^0\delta}{ml\sqrt{gR}} \tag{51}$$

Then, as it is easy to see, system of equations (50) can be

replaced by the system of two equations

$$\frac{d\varkappa}{dt} + i\nu\varkappa = i\omega\mu, \qquad \frac{d\mu}{dt} + i\nu\mu = i\omega\varkappa \tag{52}$$

This system, in turn, is broken down into two independent equations

$$\begin{gathered}\frac{d}{dt}(\varkappa + \mu) + i(\nu - \omega)(\varkappa + \mu) = 0 \\ \frac{d}{dt}(\varkappa - \mu) + i(\nu + \omega)(\varkappa - \mu) = 0\end{gathered} \tag{53}$$

which are immediately integrated. We will have

$$\begin{gathered}\varkappa + \mu = (\varkappa_0 + \mu_0)\exp\left(-i\int_0^t(\nu - \omega)\,dt\right) \\ \varkappa - \mu = (\varkappa_0 - \mu_0)\exp\left(-i\int_0^t(\nu + \omega)\,dt\right)\end{gathered} \tag{54}$$

where $\varkappa_0$ and μ_0 are the initial values of functions $\varkappa(t)$ and $\mu(t)$ at point in time t = 0.

Using formulae (54) and (51), it is already easy to also represent the unknown varibles α, β, γ and δ in explicit form as functions of their initial values α_0, β_0, γ_0, δ_0 and time t.

With respect to variables $\alpha/\sqrt{gR}$, β, γ and $2B\sin\varepsilon^\circ/ml\sqrt{gR}$, where, according to relationship (45), ε° is determined by the formula

$$\sin\varepsilon^\circ = \sqrt{1 - \left(\frac{mlv}{2B}\right)^2}, \tag{55}$$

system of equations (50) is reduced to a system of linear differential equations with constant coefficients, as long as the angular velocity ω is constant. The latter is expressed by the formula

$$\omega = U \sin\varphi + \frac{V_E}{R} \operatorname{tg}\varphi - \frac{d\psi}{dt} \quad (56)$$

where ψ is the true course of the ship, i.e, the angle between its velocity vector (to be more precise, the velocity of the suspension point of the framework) and the meridian of the location counted clockwise from the direction to the north.

7. The above theory of small movements of the gyroscope framework around the mobile axes of the Darboux triangle related to the trajectory of the suspension point leads to nonattenuating oscillations. The problem of the strict substantiation of the stability of undisturbed movement defined by nonlinear equations (40) requires additional study, of course.

The introduction of attenuation similar to that used in ordinary two-gyroscope compasses into the mechanical system of the gyroframe leads to the manifestation of ballistic deviations, i.e., additional deviations of the variables α, β, γ and δ caused by the law of the change in the acceleration of the suspension point during its

movement over sphere S. The estimation of these deviations also requires special consideration. Finally, in the future it will be important to determine the distortions introduced into the movements of the gyroscope framework in question by the difference of the Earth's shape from a sphere.

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